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Vibrations of a quantized vortex in a weakly interacting Bose fluid

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Abstract. The vibrations of a quantized vortex are studied in the long wavelength limit. It is shown that the frequency is proportional to the square of the wavenumber in contradistinction to previous work.

1. Introduction

It is well known that vortex lines can exist in superfluid helium and that these lines can be made to vibrate (see eg the review article by Vinen (1961)). To study theoretically the properties of vortices and in particular their possible modes of vibration it is necessary to consider a model equation which describes the relevant properties of ^{4}HeII.

A model equation which has received a considerable amount of attention is one based on Bogoliubov's theory of a weakly interacting Bose gas. In this model the properties of the system are described by a macroscopic wavefunction ψ which satisfies the nonlinear Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_0}\nabla^2\psi + V_0|\psi|^2\psi. \tag{1.1}$$

V_0 is a constant and is a measure of the strength of interaction between the particles. $|\psi|^2$ is interpreted as the condensate density, which for a weakly interacting gas is approximately equal to the total density. Unfortunately it is found that the condensate density is only of order 20% of the total density even as one approaches the absolute zero of temperature (Cummings *et al* 1970), so the weakly interacting assumption basic to the derivation of (1.1) is not strictly valid for helium. However, there seems at present to be no alternative equation which is at all tractable. Thus in this paper we will discuss the solutions of (1.1) and expect the qualitative features to apply to helium. Various other authors, in particular Pitaevskii (1961) and Fetter (1965, 1972), have used essentially equation (1.1) to study vortex vibrations.

An equation for the equilibrium state of a single line vortex is obtained by writing

$$\psi(x, t) = \Phi_e(r)\exp(\mathrm{i}\theta - \mathrm{i}Et/\hbar)$$

and substituting in (1.1) to give

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\Phi_e}{\mathrm{d}r}\right) + \left(\alpha - \frac{1}{r^2}\right)\Phi_e - V\Phi_e^3 = 0, \tag{1.2}$$

where $\alpha = 2m_0E/\hbar^2$, $V = 2m_0V_0/\hbar^2$. No exact solution of (1.2) is known but a useful approximation is one originally given by Fetter (1965), namely

$$\Phi_e^2(r) = \frac{\alpha^2 r^2}{4V+\alpha r^2}.$$

This has the correct behaviour as $r \to 0$ and $r \to \infty$.

Pitaevskii (1961) has studied small amplitude oscillations of a vortex line and showed that for long wavelength disturbances the frequency ω and wavenumber k (along the axis of the vortex) are related in the following manner:

$$\omega = \frac{\hbar k^2}{4m_0}\ln\left(\frac{\alpha}{k^2}\right).$$

The k dependence is identical to that obtained for the vibrations of a classical vortex by Kelvin many years ago (Thomson 1910). Fetter (1965) has studied the mathematical properties of the equations describing the vibrations of a quantized vortex and has shown that there exists a continuum of modes which he identifies with scattering states, and possibly an infinite number of bound states, the lowest of which he identifies with the mode discussed by Pitaevskii. Fetter uses a variational principle to investigate the ω, k relation for this mode for all k, but unfortunately his method breaks down in the limit $k \to 0$ so he is unable to make an independent judgement on the validity of Pitaevskii's result.

It is the purpose of the present paper to study the long wavelength vibrations of a line vortex by applying a perturbation method, previously developed by the author to study the stability of nonlinear waves (Rowlands 1969), to equation (1.1). It is found that there exists discrete states, with $\omega = bk^2$ and continuum states with $\omega = \lambda + ak^2$. Here a and b are constants but λ can take a continuum of values.

An inconsistency is shown to exist in the expansion scheme used by Pitaevskii. The difference between the above results and that of Kelvin is attributed to the fact that in a classical vortex the density is discontinuous at the core radius whilst for a quantum vortex the density is a continuous function of the radial distance.

2. Vibrations of a line vortex

Equations describing the small amplitude oscillations of a vortex are obtained by linearizing equation (1.1) about the equilibrium state (1.2). It is first, however, convenient to write

$$\psi(x, t) = \Phi(x, t)\exp(\mathrm{i}\chi(x, t)),$$

where Φ and χ are both real, write

$$\Phi(x, t) = \Phi_e(r) + \delta\Phi(r)\exp(\mathrm{i}m\theta + \mathrm{i}kz - \mathrm{i}\omega t),$$

$$\chi(x, t) = \theta - Et/\hbar - \mathrm{i}\delta\bar{\chi}\exp(\mathrm{i}m\theta + \mathrm{i}kz - \mathrm{i}\omega t),$$

and treat $\delta\Phi$ and $\delta\bar{\chi}$ as small.

The $m = 0$ case has to be treated separately. For $m \neq 0$ the linearized equations may be written in the form

$$L_1\delta\Phi + \frac{2}{r^2}\delta\chi = k^2\delta\Phi - \bar{\omega}\delta\chi, \tag{2.1}$$

$$L_2\delta\chi + \frac{2m^2}{r^2}\delta\Phi = k^2\delta\chi - \bar{\omega}m^2\delta\Phi, \tag{2.2}$$

where $\delta\chi = -m\Phi_e\delta\tilde{\chi}$ and $\bar{\omega} = (2m_0/\hbar)\omega/m$. Here

$$L_1 = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} + \alpha - 3V\Phi_e^2 - \left(\frac{1+m^2}{r^2}\right),$$

and

$$L_2 = L_1 + 2V\Phi_e^2.$$

These are equivalent to equation (22) of Pitaevskii (1961) and equation (18) of Fetter (1965). The above equations, together with the condition that $\delta\Phi$ and $\delta\chi$ must be bounded functions of r, serve to define the relation between ω and k.

A major difficulty in trying to solve (2.1) and (2.2) is that the r dependence of the coefficients, through Φ_e, is complicated and in fact is only known numerically. However, by using a perturbation theory developed by the present author (Rowlands 1969), one can, in the long wavelength limit, obtain the desired ω, k relationship without a detailed knowledge of the r dependence of Φ_e.

First consider solutions of (2.1) and (2.2) in the limit $\omega = k = 0$, namely $\delta\Phi_0$ and $\delta\chi_0$ respectively. For $r \to 0$ both $\delta\Phi_0$ and $\delta\chi_0$ have two solutions proportional to $r^{|m-1|}$ and $r^{|m+1|}$, whilst the other two solutions are unbounded. For $r \to \infty$ two types of solution exist; one where $\delta\Phi_0 \simeq 1/r^{|m|+2}$ and $\delta\chi_0 \simeq 1/r^{|m|}$, and the other where both $\delta\Phi_0$ and $\delta\chi_0$ are proportional to $\exp\{-\sqrt{(2\alpha)}r\}/\sqrt{r}$. Thus there exists two distinct types of solution which will be referred to as class I and II solutions. Both are finite as $r \to 0$, but class I solutions are those where $\delta\Phi_0 \to 1/r^{|m|+2}$ and $\delta\chi_0 \to 1/r^{|m|}$ as $r \to \infty$ whilst class II solutions are those where both $\delta\Phi_0$ and $\delta\chi_0$ approach zero exponentially as $r \to \infty$.

The perturbation method is based on application of the so-called many-time perturbation method used in statistical mechanics (see eg Frieman 1963) which itself is based on a method due to Bogoliubov and Krylov (Bogoliubov and Mitropolsky 1961). In the long wave limit ($k \to 0$) we write

$$\delta\Phi(r) = A(r_1, r_2, \ldots)(\delta\Phi_0(r) + k\delta\Phi_1(r, r_1, \ldots) + k^2\delta\Phi_2 + \ldots),$$

$$\delta\chi(r) = A(r_1, r_2, \ldots)(\delta\chi_0(r) + k\delta\chi_1(r, r_1, \ldots) + k^2\delta\chi_2 + \ldots),$$

and

$$\bar{\omega} = 0 + k\bar{\omega}_1 + k^2\bar{\omega}_2 + \ldots.$$

The quantities r_1, r_2, etc are defined such that $\mathrm{d}r_1/\mathrm{d}r = k$, $\mathrm{d}r_2/\mathrm{d}r = k^2$, etc but otherwise these variables are treated as independent. To first order in the expansion scheme

$$L_1\delta\Phi_1 + \frac{2}{r^2}\delta\chi_1 = -\bar{\omega}_1\delta\chi_0 - \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r_1}\left(2\frac{\mathrm{d}\delta\Phi_0}{\mathrm{d}r} + \frac{\delta\Phi_0}{r}\right), \tag{2.3}$$

$$L_2\delta\chi_1 + \frac{2m^2}{r^2}\delta\Phi_1 = -\bar{\omega}_1 m^2\delta\Phi_0 - \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r_1}\left(2\frac{\mathrm{d}\delta\chi_0}{\mathrm{d}r} + \frac{\delta\chi_0}{r}\right), \tag{2.4}$$

with L_1 and L_2 operating with respect to r only. It is now necessary to consider the adjoint equations in the limit $\varpi = k = 0$. These take the form

$$L_1\delta\Phi_0^+ + \frac{2m^2}{r^2}\delta\chi_0^+ = 0$$

and

$$L_2\delta\chi_0^+ + \frac{2}{r^2}\delta\Phi_0^+ = 0.$$

Comparison of these equations with those for $\delta\Phi_0$ and $\delta\chi_0$ (namely equations (2.1) and (2.2) with $\varpi = k = 0$) show that $\delta\Phi_0^+ = m^2\delta\Phi_0$ and $\delta\chi_0^+ = \delta\chi_0$.

Multiplication of (2.3) by $r\delta\Phi_0^+$, (2.4) by $r\delta\chi_0^+$, integrating over all r and adding gives the consistency equation

$$\varpi_1\{2\langle\delta\Phi_0, \delta\chi_0\rangle m^2\} = -\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r_1}\left(m^2\left\langle \delta\Phi_0, 2\frac{\mathrm{d}\delta\Phi_0}{\mathrm{d}r}+\frac{\delta\Phi_0}{r}\right\rangle + \left\langle \delta\chi_0, 2\frac{\mathrm{d}\delta\chi_0}{\mathrm{d}r}+\frac{\delta\chi_0}{r}\right\rangle\right),$$

where

$$\langle X, Y\rangle = \int_0^\infty r\,\mathrm{d}rXY.$$

Now

$$\left\langle \delta\Phi_0, 2\frac{\mathrm{d}\delta\Phi_0}{\mathrm{d}r}+\frac{\delta\Phi_0}{r}\right\rangle = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}(r\delta\Phi_0^2)\,\mathrm{d}r = 0,$$

since $\delta\Phi_0$ remains finite as $r \to 0$ and is of order $1/r^{|m|+2}$ as $r \to \infty$. Similarly the term involving $\delta\chi_0$ in the coefficient of $A^{-1}(\mathrm{d}A/\mathrm{d}r_1)$ is zero so that the consistency condition gives $\varpi_1 = 0$. It may then be shown using equations (2.3) and (2.4) that

$$\delta\Phi_0 = -\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r_1}r\delta\Phi_0,$$

and

$$\delta\chi_1 = -\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r_1}r\delta\chi_0.$$

To next order in the expansion scheme

$$L_1\delta\Phi_2 + \frac{2}{r^2}\delta\chi_2$$
$$= -\varpi_2\delta\chi_0 + \delta\Phi_0 - \frac{1}{A}\frac{\mathrm{d}}{\mathrm{d}r_1}\left\{A\left(2\frac{\mathrm{d}\delta\Phi_1}{\mathrm{d}r}+\frac{\delta\Phi_1}{r}\right)\right\} - \frac{1}{A}\frac{\mathrm{d}^2A}{\mathrm{d}r_1^2}\delta\Phi_0$$
$$-\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}r_2}\left(2\frac{\mathrm{d}\delta\Phi_0}{\mathrm{d}r}+\frac{\delta\Phi_0}{r}\right),$$

and a similar equation is obtained for $\delta\chi_2$. A consistency condition may then be obtained and by using the above forms for $\delta\Phi_1$ and $\delta\chi_1$ it takes the form

$$-\varpi_2 2m^2\langle\delta\Phi_0, \delta\chi_0\rangle + \gamma\left(1 - \frac{1}{A}\frac{\mathrm{d}^2A}{\mathrm{d}r_1^2}\right) + \frac{1}{A}\frac{\mathrm{d}^2A}{\mathrm{d}r_1^2}\left(m^2\left\langle \delta\Phi_0, 3\delta\Phi_0 + 2r\frac{\mathrm{d}\delta\Phi_0}{\mathrm{d}r}\right\rangle\right.$$
$$\left. + \left\langle \delta\chi_0, 3\delta\chi_0 + 2r\frac{\mathrm{d}\delta\chi_0}{\mathrm{d}r}\right\rangle\right) = 0,$$

where $\gamma = m^2\langle\delta\Phi_0^2\rangle$. This equation may be simplified by integration by parts to give finally

$$(-\bar{\omega}_2 2m^2\langle\delta\Phi_0, \delta\chi_0\rangle + \gamma)A + \beta\frac{d^2A}{dr_1^2} = 0,$$

where $\beta = \lim_{r\to\infty} r^2\delta\chi_0^2$.

Now as stated above there are two classes of solution depending on the r dependence of $\delta\Phi_0$ and $\delta\chi_0$ as $r \to \infty$. We see that for class II the quantity β as defined above is zero so that

$$\omega = \frac{\hbar}{2m_0}m\bar{\omega}_2k^2 = \frac{\hbar}{2m_0}k^2\left(\frac{m^2\langle\delta\Phi^2\rangle + \langle\delta\chi_0^2\rangle}{2m\langle\delta\Phi_0, \delta\chi_0\rangle}\right), \tag{2.5}$$

that is a discrete state for a particular value of k. These correspond to bound states as defined by Fetter since the solutions decrease exponentially as $r \to \infty$. This same result follows from a variational principle applied to (2.1) and (2.2) if for the trial functions one takes $\delta\Phi = \delta\Phi_0$ and $\delta\chi = \delta\chi_0$.

For class I type solutions it is seen that $\beta = 0$ for $|m| > 2$ so again we only have a discrete solution but now the solutions do not approach zero exponentially as $r \to \infty$ but rather as powers of $1/r$. The case $|m| = 1$ must be treated separately. In particular it is found by differentiation of (1.4) that $\delta\Phi_0 = d\Phi_e/dr$ and $\delta\chi_0 = \Phi_e/r$, in which case the equation for A reduces to

$$\frac{d^2A}{dr_1^2} + (\gamma' - \bar{\omega}_2)A = 0,$$

where

$$\gamma' = \frac{V}{\alpha}(\alpha\langle\Phi_e^2\rangle - V\langle\Phi_e^4\rangle).$$

Thus

$$\omega = \lambda + m\gamma' k^2\left(\frac{\hbar}{2m_0}\right), \tag{2.6}$$

where λ may take a continuum of values and is associated with the r_1 dependence of A.

In summary it has been shown that a discrete solution to the equations exists with $\omega = ak^2$ where a is a constant given by (2.5). Further for $|m| = 1$ there also exists solutions where $\omega = \lambda + bk^2$ with λ taking a continuum of values and b given by (2.6).

The case of $|m| = 1$ has been studied by Pitaevskii who shows in contrast to the above that a discrete state exists with

$$\omega = \frac{\hbar k^2}{2m_0}\ln\left(\frac{1}{kr_0}\right).$$

If one assumes that this result is correct then in equation (22) of Pitaevskii one may neglect the term proportional to χ^2 since $\epsilon = -\chi^2 \ln \chi$. If these equations are then linearized about the state $\epsilon = 0$ one obtains Pitaevskii's equations (25). If the first of these equations is multiplied by ξf^0, integrated over all ξ, and added to the second equation multiplied by ξf_1^0 and integrated, one obtains the consistency condition

$$\int_0^\infty \xi\{(f_1^0)^2 - (f^0)^2\}\, d\xi = 0.$$

However, using equation (23) of that paper one finds that this integral is simply equal to $-\lim_{\xi\to\infty} \Psi_0^2(\xi) = -1$. Thus there is an inconsistency in the expansion scheme, which in the light of the present work is seen to be associated with the ϵ (or ω) dependence on k. Of course a dispersion relation of the form obtained by Pitaevskii was obtained many years ago by Kelvin for a classical vortex, using a nonperturbative analysis. However, in the classical problem the structure of the vortex has to be assumed and Kelvin took it to be of a form where the density gradient had a discontinuity at the so-called vortex core radius. It is this discontinuity which leads to the particular form of the dispersion relation. In the present problem the structure of the core is not arbitrary, but given by Φ_e, and in particular is continuous, so that the result that the dispersion relation is qualitatively different from Kelvin's is not surprising.

Fetter has used a variational method to discuss the discrete state corresponding to $m = 1$. Unfortunately his method breaks down for $k \to 0$ as is seen from table 2 of Fetter where $\kappa(\equiv k)$ becomes imaginary in this limit. This difficulty is probably due to the fact that the trial functions Fetter used do not have the correct r dependence for $r \to \infty$. This is shown by the fact that if the trial functions are taken to be $\delta\Phi_0$ and $\delta\chi_0$ then one does get the correct ω, k relation as given by (2.5) above.

As stated above the $m = 0$ case has to be treated separately. For this case, the lowest order equations are

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\delta\Phi_0}{\mathrm{d}r}\right) + \left(\alpha - 3V\Phi_e^2 - \frac{1}{r^2}\right)\delta\Phi_0 = 0, \tag{2.6}$$

and

$$\frac{\Phi_e}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\delta\chi_0}{\mathrm{d}r}\right) + 2\frac{\mathrm{d}\Phi_e}{\mathrm{d}r}\frac{\mathrm{d}\delta\chi_0}{\mathrm{d}r} = 0. \tag{2.7}$$

One solution of the last equation is $\delta\chi_0 = $ constant. By considering the small r expansion of this equation it is found that this is the only solution which remains finite for all r. To study the solution of (2.6), first consider the eigenvalue problem

$$\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} + \alpha - V\Phi_e^2 - \frac{1}{r^2}\right)\chi_n = \lambda_n\Phi_e^2\chi_n$$

and note that one solution is known namely $\chi = \Phi_e, \lambda = 0$. It follows from the general theory of differential equations (Ince 1927) that since Φ_e is monatomic that the other eigenfunctions which exist do so for $\lambda_n < 0$. Thus a solution does not exist for $\lambda_n = 2V$ from which we conclude that (2.6) does not have a solution. This being so one cannot start a perturbative expansion about $\omega = k = 0$, a result which is borne out by Fetter's variational calculation which suggests that ω approaches a constant as $k \to 0$.

3. Conclusions

The nonlinear Schrödinger equation is taken as a model equation to study the properties of vortices in helium II. A perturbation expansion is used to obtain solutions of this equation corresponding to the long wavelength vibration of vortices. Two types of solutions are found; those where $\omega = ak^2$ which correspond to discrete states, and those where $\omega = \lambda + bk^2$, where λ can take a continuum of values (a and b are constants). This result is in disagreement with that of Pitaevskii who found ω to be proportional to $k^2 \ln k$.

It is clear that the method presented in this paper could be used to obtain the higher order terms in the series expansion for ω. Unfortunately it is extremely difficult, probably impossible, to make a statement about the convergence of the method so the results obtained in the paper rest on the assumption that the series solution is at least asymptotic.

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